EXTRAITS

Chaouqi Misbah

Complex Dynamics and Morphogenesis

An introduction to nonlinear science

Springer

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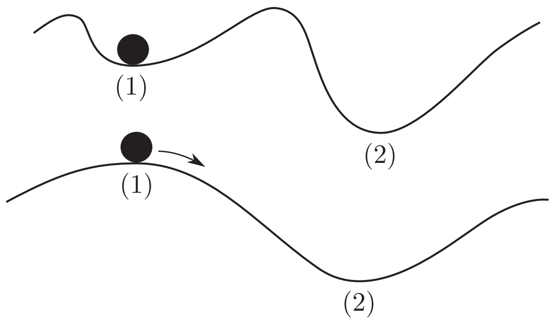

Figure 1.2 – In the top figure, a ball at point (1) has no chance of finding itself at point (2) due to any natural fluctuation. We call this position linearly stable. To change from point (1) to (2) a strong outside perturbation is necessary. Thus, position (1) is linearly stable but nonlinearly unstable. In the bottom figure, the ball is in an unstable position. In this case a small perturbation is enough to cause the transition from (1) to (2), and position (1) is linearly unstable.

The general and universal features of nonlinear phenomena can be represented in two different languages: catastrophe and bifurcation theory. Catastrophe theory deals with the determination of the geometric figure tracing out the boundaries of qualitatively differing solutions in parameter space (figure 1.1). Bifurcation theory deals instead with the specification of the evolution of one dynamic variable of the system (such as the equilibrium position of a mass in a mechanics problem) as a function of the control parameters (see figure 1.3, a bifurcation diagram).

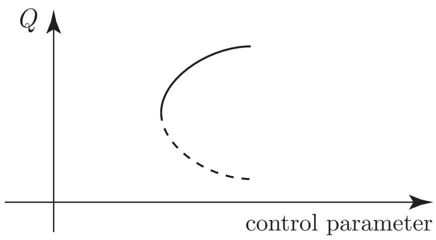

Figure 1.3 – A bifurcation diagram, representing a quantity Q characteristic of the nonlinear system studied as a function of a control parameter. The broken line represents an unstable branch, and the continuous line the stable branch.

Though two different representations, catastrophe and bifurcation describe the same reality; and especially, both search for universal behaviors independent of any specific model studied (whether it be a chemical, biological, or economic system, etc.). The goal of catastrophe theory is to determine the

Bifurcations are often accompanied by a loss of symmetry. A bed of sand is flat before it becomes sculpted by ripples, so the topography from any two points is indistinguishable. As ripples form, the topography between crest and trough becomes starkly distinguished. Now one must travel a distance equal to the repetition length of the motif (the wavelength of the ripples) to find the same landscape (such as a crest). This is an example of translation symmetry breaking; after the formation of the folds, the symmetry has become discrete, and so one must travel an integer number of wavelengths to find the same topography.

In the neighborhood of a bifurcation point (i.e. when the value of the control parameter is close enough to the critical value needed to give rise to order), all systems are described by a universal equation called the *amplitude equation*. Once a system is far from thermal equilibrium (for example, if we add more and more energy), the ordered state can, in turn, become unstable and undergo a secondary bifurcation. Each new bifurcation leads to a new loss of symmetry. In general, the final state of each system is temporal and spatial disorder, namely spatio-temporal chaos or turbulence: the system then loses all temporal and spatial correlations.

A physicist may be tempted to draw parallels between primary bifurcations (the emergence of order) and the theory of phase transitions (for example, that of the liquid-solid transition occurring at a low enough temperature). We will see that this is a valid analogy so long as the system is studied in the neighborhood of the instability threshold, and if the instability is stationary (i.e. has no time oscillation). This analogy quickly reaches its limits, and non-equilibrium processes come to the fore, taking the studied systems to new depths. The dynamics become complex, and have no resemblance to phase transitions. Non-equilibrium systems may have a wide variety of dynamic behaviors – something not allowed for by the thermodynamics of a system globally in equilibrium (see chapter 12).

Systems in equilibrium[8] are governed by principles that are generally well-established: for example, they are characterized by a minimum value of energy, or a maximum of entropy, according to the exact configuration considered (e.g. in thermodynamics, processes having to do with a constant temperature or a constant pressure are evoked). These principles lead to what we call variational models[9]. Systems that are not in equilibrium are non-variational, by virtue of the fact that their state, or their dynamics, cannot

8. In physics, we may speak of thermodynamic equilibrium, but the notion of non-equilibrium is more general, as we have mentioned before.

9. This name references the fact that the state that is selected is either a minimum or a maximum of a certain quantity, such as energy. The extremum is obtained by taking the zero of the derivative of a certain quantity, such as energy (or of a functional, that is, a function of a function).

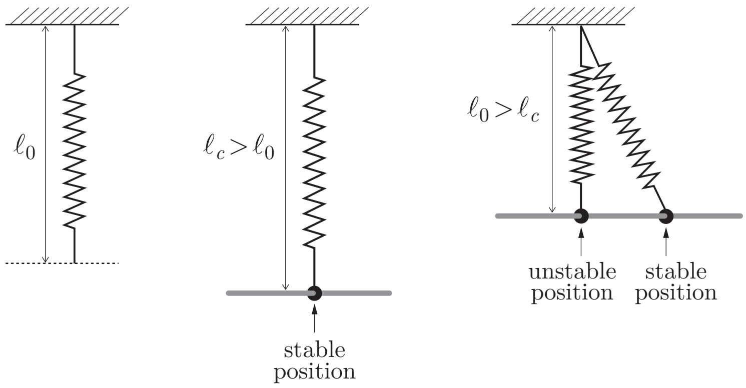

Figure 2.3 – A schematic view of different configurations. From left to right: rest length; stretched spring in vertical position (the spring resists moving away due to increased stretching); and compressed spring in vertical position (a deviation from that position is favorable since it reduces stretching).

2.1.3. Analysis of the Bifurcation

So, the $x = 0$ solution of the system studied in this chapter (see figure 2.1) becomes unstable as soon as the imposed length of the spring at the vertical position ℓ_c is smaller than its rest length ($\ell_c < \ell_0$). Let us now determine the new equilibrium position when in the vicinity of the instability threshold[2] i.e. $\ell_c \simeq \ell_0$. Once the threshold $\ell_c = \ell_0$ is reached, the mass displaces from its vertical position $x = 0$ and the nonlinear effects of the higher order terms (x^4) intervene to ensure saturation.

Using the condition $dE_p/dx = 0$ to define the extremas of energy and their stability relative to the potential energy, we determine the equilibrium solutions: (i) when $\ell_c > \ell_0$, solution $x = 0$ is stable; (ii) whereas when $\ell_c < \ell_0$, solution $x = 0$ is unstable (see figure 2.2). From the fourth order Taylor expansion of the potential energy (equation (2.4)) we find the following solutions (subscript 0 in x_0 indicates an equilibrium solution):

$$x_0 = 0 \quad \text{and} \quad x_0 = \pm \ell_0 \sqrt{2\frac{(\ell_0 - \ell_c)}{\ell_c}}. \tag{2.5}$$

2. Though the system studied in this chapter can be determined entirely with an analytic calculation, most nonlinear systems cannot. It is thus necessary to take up linear expansions to analyze the system. Focusing on the vicinity of the instability threshold, besides justifying the Taylor expansions of functions related to the system, gives the problem a universal character, independent of the nature of the system and thus of the specific mechanisms that govern it.

The first solution, $x_0 = 0$, always exists, whereas the second solution requires that $\ell_c < \ell_0$. When $x_0 = 0$ becomes unstable ($\ell_c < \ell_0$) the new double solution $x_0 = \pm\ell_0\sqrt{2(\ell_0 - \ell_c)/\ell_c}$ takes over. We say there is a bifurcation of the stationary solution $x_0 = 0$ toward a pair of stationary solutions $x_0 = \pm\ell_0\sqrt{2(\ell_0 - \ell_c)/\ell_c}$.

Plotting the value of x_0 as a function of the spring length at rest ℓ_0, which in this case is our control parameter, we obtain what is called a bifurcation diagram (see figure 2.4), which is simply the portrait of the stationary solutions (equation (2.5)). The broken curve on figure 2.4 represents the unstable solution. This is the usual convention in bifurcation theory. Let us mention that the bifurcation in question, called pitchfork bifurcation because of the shape of the curve shown in figure 2.4, is sometimes also called a *supercritical bifurcation* (see section 3.3). Furthermore, it belongs to the family of *cusp* catastrophes, which we will discuss further in chapter 4.

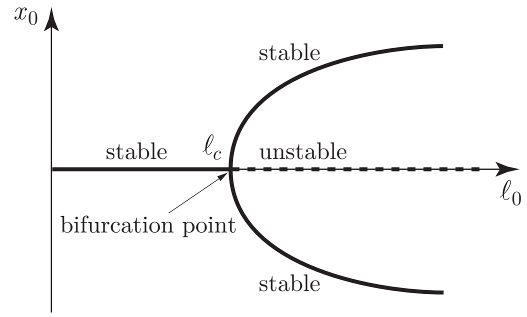

Figure 2.4 – A pitchfork bifurcation diagram. Continuous lines correspond to stable branches, and the discontinuous line to unstable branches.

2.1.4. Universality in the Vicinity of a Bifurcation Point

In the study of any nonlinear system, the first major steps are to search for bifurcation points and to look at polynomial expansions in the vicinity of these points.

The full potential energy $E_p(x)$ (equation (2.1)) is simple enough that we could solve it without using a polynomial approximation. It can easily be checked that the qualitative behavior found with the fourth order approximation (see figure 2.2) is equivalent to the behavior found by analysis of the exact expression (equation (2.1)).

The resemblance in the potential energy's qualitative behavior when using both the exact and approximate forms is indicative of the fact that the qualitative change in solutions only occurs near $x = 0$; that is, once the length

While the notion of structural stability is perfectly defined and non-ambiguous in a mathematical sense, it calls for some clarification when evoked in an experimental context. For example, the bifurcation diagram of an experimental model (see figure 3.26(c)) is representative of a real experiment that is invariably skewed by measurement errors or some weak bias. The theoretical models of the pitchfork bifurcation or of the imperfect bifurcation (see figure 3.26(a,b)) would both be a priori good candidates for the experimental result so long as the criteria for distinguishing between the two is not at our disposal. Thus, the choice of a model to describe the experimentally obtained bifurcation appears to be somewhat subjective. This indeterminacy that comes from the experimental context is something inveterate to the experimental method; nonetheless it should not eliminate the need for a rigorous classification of bifurcations. This classification is found in the framework of catastrophe theory.

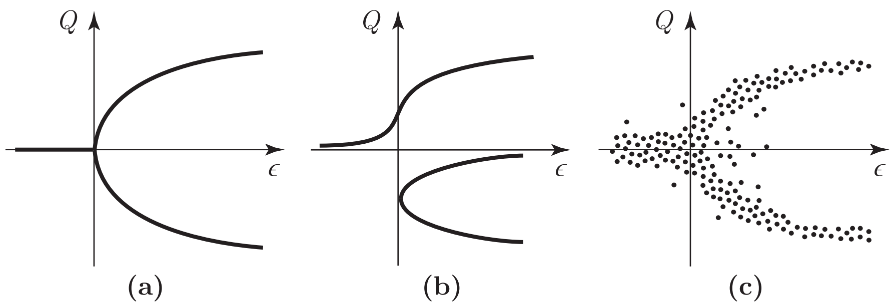

Figure 3.26 – (a) Pitchfork bifurcation; (b) imperfect bifurcation; (c) schematic representation of experimental results and their uncertainty.

3.7. What Does Catastrophe Theory Consist Of?

Although we will undertake this topic in greater detail in the following chapter, we will first introduce and illustrate catastrophe theory with the help of an already familiar example. In contrast to the study of bifurcations in which the evolution of solutions are represented in a bifurcation diagram (for example, see the imperfect bifurcation in figure 3.16 representing the evolution of an internal variable of amplitude A as a function of external control parameter ϵ), catastrophe theory studies the qualitative change in a solution's behavior solely within the parameter space of the control parameters. Furthermore, catastrophe theory relies on the notion of structural stability to determine the generic forms of different types of changes in behavior. We will

4.6. Hyperbolic Umbilic Catastrophe

As we have already emphasized for systems of a single degree of freedom, the weakest nonlinear term of the potential V is the cubic term. Making a simple generalization from the unidimensional case, we can write the two dimension singularity as $A^3 + B^3$. As was the case in the one dimensional problem, the quadratic terms A^2 and B^2 can be eliminated through a change of variables. However, since the term AB cannot be eliminated simultaneously and since the linear terms A and B must remain in the equation, as in the unidimensional case, the generic form of the potential is:

$$V = A^3 + B^3 + wAB - uA - vB. \tag{4.22}$$

Since we will need three parameters for the universal unfolding of $A^3 + B^3$, we can say that this catastrophe under consideration is of co-dimension 3. We have three conditions: $V_A = V_B = 0$ and the Hessian $\Delta = 0$. We have five unknowns: A, B, u, v and w. The two first conditions permit us to eliminate A and B in favor of (u, v, w), and the third condition provides the relationship between these three parameters from which we can plot the surface of the catastrophe which we call *hyperbolic umbilic*[11] (see figure 4.8 for plots in three dimensions and on the plane (u–v)).

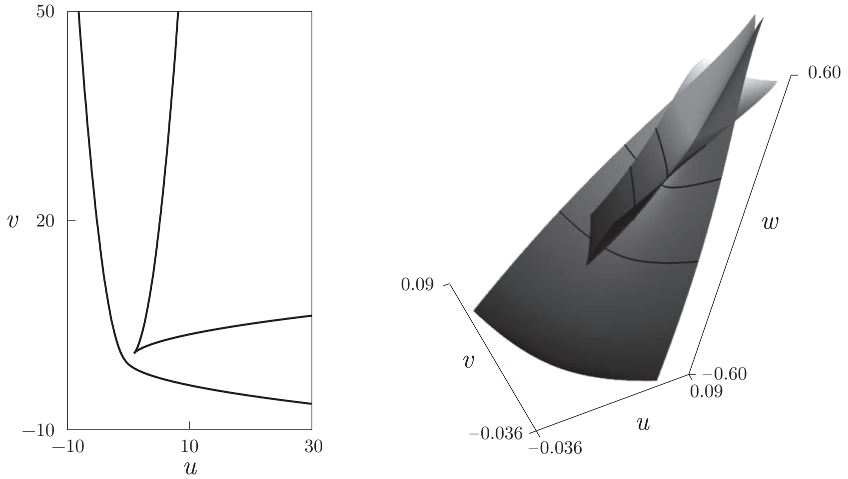

Figure 4.8 – Hyperbolic umbilic catastrophe. Left: a section corresponding to $w = 2$. Right: a representation of the catastrophe in three dimensions. [© LiPhy]

11. The word umbilic signifies a point of a curved surface where all the sections of a normal plane have the same radius of curvature. This catastrophe is sometimes called the "wave" catastrophe.

classification to rely on. Nonetheless, adopting the formal framework of bifurcations, we can procure descriptions founded on the existing symmetries of the problem.

4.11. Solved Problem

4.11.1. Umbilic Hyperbolic Catastrophe

Write the three equations which determine the umbilic hyperbolic catastrophe in vector space (u, v, w).

Solution. Starting with potential $V(A, B)$ (equation (4.22)), we obtain the fixed point equations $\partial V/\partial A = \partial V/\partial B = 0$. We take the determinant of the Hessian matrix to zero, expressing the annulment of the potential's concavity, resulting with $\Delta = V_{AA}V_{BB} - (V_{AB})^2 = 0$ (see equation (4.21)). These conditions give us the following three relations:

$$\begin{aligned} 3A^2 + wB - u &= 0, \\ 3B^2 + wA - v &= 0, \\ 36AB - w^2 &= 0. \end{aligned} \tag{4.25}$$

Suppose that parameter w is not zero ($w \neq 0$). The above equations give us the parametric equations for the two parameters u and v:

$$\begin{aligned} u &= 3A^2 + \frac{w^3}{36A}, \\ v &= \frac{3w^4}{36^2A^2} + wA. \end{aligned} \tag{4.26}$$

We fix the value of parameter w, and vary the values for A (for example, between -2 and 2). We can calculate an ensemble of pairs (u, v) for each value of A which would allow us to trace a curve in the plane (u–v) (see figure 4.8). This curve is the projection onto the plane (u–v) of the umbilic catastrophe. If w is in fact zero ($w = 0$), the third condition of annulment for the potential's concavity (equations (4.25)) implies that either variable A or B is annulled: (i) if $A = 0$, the first condition of the same system (equations (4.25)) implies $u = 0$, while the second implies $v > 0$; (ii) if $B = 0$, we find $v = 0$ and $u > 0$.

To conclude, if $w = 0$, the curve of the catastrophe is the ensemble of axes u and v. Finally, note that for small enough A, v is positive regardless of the sign of A, because the first term in $1/A^2$ of the parametric equation for v (equation (4.26)) is dominant over the second term in A, while u changes sign with A. This results in a discontinuous curve in the plane of (u–v); the curve is composed of two disjoint sections.

8.10. Critical Dimension for Attaining Chaos

Chaos is characterized by an extreme sensitivity to initial conditions. This sensitivity is illustrated by the exponential divergence of two trajectories originating from very similar initial conditions[6]; the system very quickly forgets the initial proximity of the two trajectories.

Dissipation of energy in a system confines the trajectory associated with the chaotic movement to a finite region of phase space. The trajectory is limited to the area of an attractor, called a strange attractor (see section 8.7). This condition seems a priori contradictory to the first property which is characteristic of chaos, in which *two trajectories issued from neighboring initial conditions will diverge in time.*

Now we will take a closer look. In two dimensions (a plane) it is impossible for two trajectories to grow exponentially apart: on one hand, as seen above, due to dissipation trajectories covering a limited region of the plane (attractor); on the other hand, because of determinism two trajectories originating from different initial conditions never cross. We present a proof by contradiction. If, in a phase plane, there exists a trajectory which at one point intersects with itself, and if we take as an initial condition this same point of intersection (see figure 8.14), it is impossible to determine a priori from this point which was the previous evolution of the trajectory (left or right branch). The future indeterminacy linked with the indeterminacy of the trajectory is thus in contradiction with the principle of determinism. In a chaotic regime, two initially close trajectories will diverge exponentially with time. However, due to the dense character of trajectories in a chaotic regime, a trajectory that has to deviate from that which was initially close is surrounded by many other trajectories that will repel it due to the non-crossing condition. Furthermore, a trajectory that is at the periphery of all other trajectories, cannot move far apart due to dissipation. Thus in a dynamical system with two degrees of freedom, chaos cannot take place.

6. We must specify that the trajectories may approach each other for certain periods of time before diverging again.

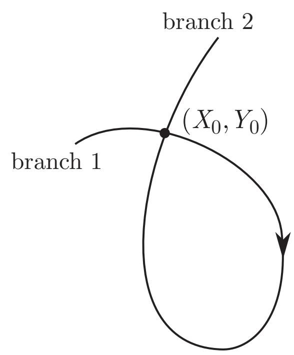

Figure 8.14 – An initial condition such as (X_0, Y_0) could belong to either branch of the trajectory. This is in contradiction with deterministic dynamics.

On the other hand, two trajectories in a 3-D phase space have the necessary room to avoid crossing each other; they can thus diverge infinitely far. The critical dimension of the phase space is in fact three – the minimum number of degrees of freedom a system needs to develop chaotic dynamics. For example, the Rössler system (see equation (8.5)) can become chaotic, while a van der Pol oscillator (see equation (5.4)) cannot; the second degree temporal evolution equation describing the van der Pol system is equivalent to two first order equations, which provides two degrees of freedom to the system and thus it is impossible for the system to go into a chaotic regime. For the van der Pol oscillator, even the scenario of a subharmonic bifurcation is impossible: all regimes which correspond to a period doubling are prohibited. The trajectory in phase space associated with this regime would need to go around twice before closing in on itself, which would be impossible without it crossing its own path, which is prohibited by the logic of determinism.

A simple pendulum (governed by a second order equation) with two degrees of freedom is subject to the same constraints as the van der Pol system. However, if one of the oscillators is driven by an external time-dependent excitation, the system gains an extra degree of freedom in the form of a temporal variable, so the pendulum may enter a chaotic regime.

To illustrate, consider the case of a sinusoidal excitation, where the pendulum equation takes the form $\ddot{\theta} + \dot{\theta} + \sin\theta = A\cos(\omega t)$, where A and ω are respectively the amplitude and the frequency of the external force. Even if the temporal variable is not a degree of freedom in the usual sense of the term, it takes care of an important condition: to determine the unique trajectory of the system. For this we need to know the details of one more variable. In this case, besides θ and $\dot{\theta}$ of a given moment, we must also know the value and sign of the force $A\cos(\omega t)$ at that moment.

9.7. Beyond the Linear Turing Instability

Throughout the chapter, we have seen that under certain general conditions, a homogeneous solution (i.e. a homogeneous mix of two chemical species) can become unstable and evolve spontaneously into a structured state (or spatially ordered) defined by a characteristic wavelength. We can study the stability of the solution using a linear analysis to start with, but need to make a nonlinear evaluation to see the whole picture. The linear study tells us that the perturbation is, at first, infinitesimal[13], and then grows exponentially with time. This means that the perturbation finishes by acquiring a large amplitude over the course of time, large enough that the linear hypothesis becomes invalid. Hence, nonlinear terms must be taken into consideration.

Then, there are two complementary nonlinear studies we can make. The first type of study (already covered in chapter 6), looks at the weakly nonlinear regime. This means making a limited expansion of equations in the neighborhood of the instability threshold. The second type of study is a direct numerical integration of the complete system of equations. This could be done with the Schnackenberg (equation (9.28)) or the Lengyel–Epstein model (equation (9.30)), among others. Numerical investigation is a general rule in the study of nonlinear systems. However, having analytic information, which is generally valid close to the instability threshold, provides an interesting basis and guide for the full nonlinear evolution. The analytic study will be seen in the next chapter.

9.8. The Diverse Turing Patterns as Found in Nature

A fascinating aspect of the Turing model is its potential for engendering many similar forms to those observed in nature, from relatively simple structures such as honeycombs or stripes (see figure 9.4), to the more complex and fascinating structures such as: (i) shell patterns (see the shells in [11]; these patterns are found by combining Turing-type equations with other chemical equations); (ii) the patterns of tropical fish (see [4]); (iii) the patterns in the fur of many animals, such as the leopard or the jaguar (see figure 9.5). These different structures emerge from a modified Turing model, and in some

13. The hypothesis of a small amplitude in the perturbation is necessary in order to legitimize the linearization of equations.

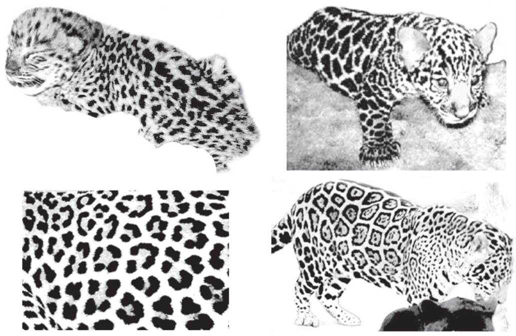

Figure 9.5 – Chemical reproductions of leopard and jaguar patterns, created with some variant of the Turing model. The top figures are real and the bottom ones were obtained with numerical simulations. The real leopard fur (top left) and the real jaguar fur (top right) present spot patterns early on in their development (of the same type as honeycomb patterns). A Turing-type model reproduces the different motifs for adults: leopard spots (bottom left) and polygons for the jaguar (bottom right). [From [62] R.T. Liu, S.S. Liaw & P.K. Maini. Two-stage Turing model for generating pigment patterns on the leopard and the jaguar, *Physical Review E*, **74**(1): 011914, 2006, © American Physical Society]

cases with the adoption of more complex geometries for the substrate (such as a curved geometry) on which the reaction takes place. The Turing model which creates the structures associated with the shell pattern or animal fur has been resolved numerically with methods that are now taught as classics in the elementary courses on numerical simulations.

Close to the instability's threshold, all the well-ordered structures – stripes, honeycombs, squares, etc. – are possible. As we go farther from the threshold, the structures can become more complex. Perfect order becomes a combination of order and disorder (or order with defects). Balancing between parameters and initial conditions allows one to reproduce images such as the ones shown in figure 9.5 (see [62]). Generally, if we begin with a Turing system, an appropriate combination of parameters (and sometimes adding a third concentration field) can also produce the different structures seen in figure 9.5, and more – such as the patterns observed on shells or tropical fish. For more details, we suggest the works which have become the inequivocal references in the domain [69, 70, 76].