EXTRAITS

Jacques Lafontaine

An Introduction to Differential Manifolds

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Chapter 1

Differential Calculus

1.1. Introduction

In this chapter, we review and reinforce the basics of differential calculus in preparation for our subsequent study of manifolds.

The majority of the concepts and results studied are generalization of concepts and results from linear algebra. We have a veritable dictionary:

smooth function	—	linear map
local diffeomorphism	—	invertible linear map
submanifold	—	vector subspace

It's necessary to understand and make this dictionary explicit.

1.1.1. What Is Differential Calculus?

Roughly speaking, a function defined on an open set of Euclidean space is differentiable at a point if we can approximate it in a neighborhood of this point by a linear map, which is called its differential (or total derivative). This differential can be of course expressed by partial derivatives, but it is the differential and not the partial derivatives that plays the central role.

The basic result, aptly called the "chain rule" assures that the differential of a composition of differentiable functions is the composition of differentials. This result gives, amongst other things, a convenient and transparent way to compute partial derivatives of compositions, but for us this will not be essential.

A fundamental notion is that of a *diffeomorphism*. By this we mean a differentiable function that admits a differentiable inverse. By the chain rule, the differential at every point of a diffeomorphism is an invertible linear map.

HEREAFTER, UNLESS OTHERWISE MENTIONED,
WE ASSUME ALL MAPS ARE SMOOTH

1.5. Submanifolds

1.5.1. Basic Properties

Intuitively, a submanifold of dimension p in $\mathbf{R}^n$ is a union of small pieces each of which can each be straightened in a way to form open subsets of $\mathbf{R}^p$. One can convince oneself for a circle that two pieces are necessary (and sufficient!).

Definition 1.20. *A subset $M \subset \mathbf{R}^n$ is a p-*dimensional submanifold *of $\mathbf{R}^n$ if for all x in M, there exists open neighborhoods U and V of x and 0 in $\mathbf{R}^n$ respectively, and a diffeomorphism*

$$f: \ U \longrightarrow V \ \text{ such that } \ f(U \cap M) = V \cap (\mathbf{R}^p \times \{0\}).$$

We then say that M is of codimension $n-p$ *in $\mathbf{R}^n$.*

This definition is better understood with Figure 1.3 kept in mind. We note that p is unique, in other words that M is not a manifold of dimension $p_1 \neq p$. The verification of this is left to the reader, unless they cannot wait until they read the next chapter, where this question will be elucidated in a more general setting.

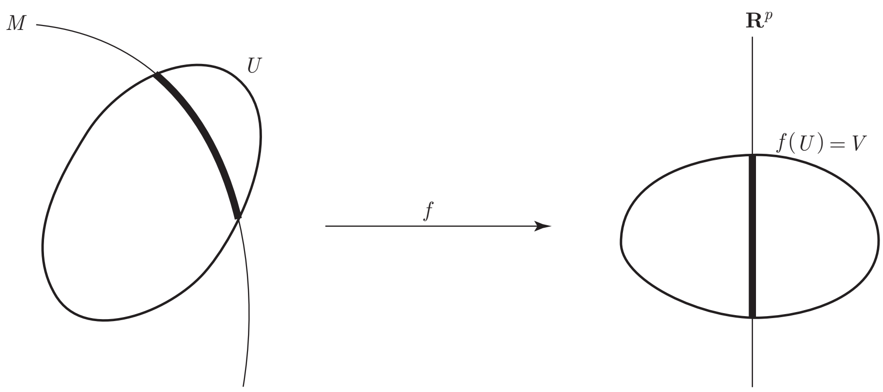

Figure 1.3: Submanifold

Remark. In this definition we can of course replace 0 and $\mathbf{R}^p \times \{0\}$ by any point and any affine subspace of dimension p.

Proposition 2.3. *Let $M \subset \mathbf{R}^n$ be a submanifold of dimension p, and let (Ω_1, g_1) and (Ω_2, g_2) be two parametrizations.*

Then

$$g_2^{-1} \circ g_1 : \ \Omega_1 \cap g_1^{-1}\big(g_2(\Omega_2)\big) \longrightarrow \Omega_2 \cap g_2^{-1}\big(g_1(\Omega_1)\big)$$

is a diffeomorphism.

PROOF. Let $m \in g_1(\Omega_1) \cap g_2(\Omega_2)$ (there is nothing to show if this intersection is empty). By Definition 1.20 there exists an open subset U containing m and a diffeomorphism f from U to $\mathbf{R}^n$ such that $f(U \cap M) = f(U) \cap (\{0\} \times \mathbf{R}^p)$. Then $f \circ g_1$ and $f \circ g_2$ are immersions from Ω_1 and Ω_2 to $\mathbf{R}^n$. Now if we consider these maps as maps with values in $\mathbf{R}^p$, we obtain smooth homeomorphisms with invertible differentials, and therefore these maps are diffeomorphisms. The same argument applies to

$$(f \circ g_2)^{-1} \circ (f \circ g_1) = g_2^{-1} \circ g_1. \qquad \square$$

As is often the case in mathematics, we take a property verified in its natural setting and elevate it to an axiom.

Definitions 2.4

a) *Two charts (U_1, φ_1) and (U_2, φ_2) of a topological manifold M are* compatible to order k *($1 \leqslant k \leqslant \infty$) if $U_1 \cap U_2 = \emptyset$ or if the map*

$$\varphi_2 \circ \varphi_1^{-1} : \ \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$$

(called a transition function*) is a C^k diffeomorphism.*

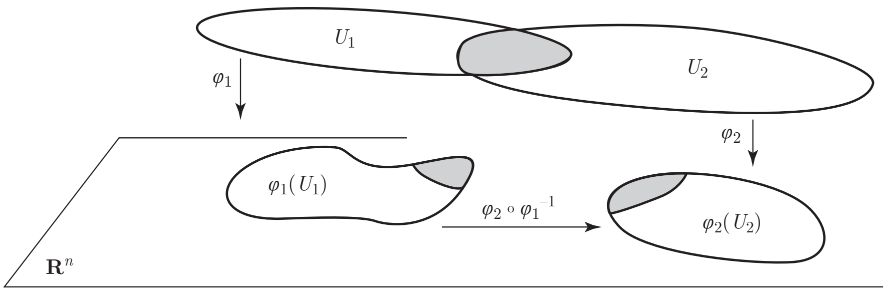

Figure 2.1: Transition function

b) *A C^k* atlas *of a topological manifold M is an atlas $(U_i, \varphi_i)_{i \in I}$ of M such that any two charts are compatible to order k.*

Take for example a smooth submanifold of codimension 1 in $\mathbf{R}^n$, defined by a submersion $f : \mathbf{R}^n \to \mathbf{R}$. This submanifold admits a smooth atlas

3.5. The Tangent Bundle

3.5.1. The Manifold of Tangent Vectors

On a manifold, as we have seen, the notion of derivation makes sense. Under these conditions, we would like to have analogous result to Theorem 3.11 for derivations at a point: a derivation on a manifold M should allow us to associate to each point m in M a tangent vector X_m of T_mM, with this correspondence being smooth in a sense that we will make precise. To do this, we will show that the set of tangent vectors is itself a manifold in a natural way. We first set

$$TM = \coprod_{m \in M} T_mM.$$

For the moment, TM is the disjoint union of different tangent vector spaces to M, without a topology. For each chart (U, φ), the map

$$\Phi : \quad (x, \xi) \longmapsto (\varphi(x), T_x\varphi \cdot \xi)$$

is a bijection from TU to $\varphi(U) \times \mathbf{R}^n$.

Given an atlas $(U_i, \varphi_i)_{i \in I}$ of M, we equip TM with a topology by imposing the following conditions:

1) the sets TU_i are open subsets of TM;

2) the maps Φ_i are homeomorphisms.

Then $\Omega \subset TM$ is open if and only if $\Phi_i(\Omega \cap TU_i)$ is an open subset of $\varphi(U_i) \times \mathbf{R}^n$ for every i. To see that these conditions are consistent, we remark that by the same definition of tangent space, if $U_i \cap U_j \neq \emptyset$, the map

$$\Phi_i \circ \Phi_j^{-1} : \ \varphi_j(U_i \cap U_j) \times \mathbf{R}^n \longrightarrow \varphi_i(U_i \cap U_j) \times \mathbf{R}^n$$

given by

$$(y, v) \longmapsto \big((\varphi_i \circ \varphi_j^{-1})(y), T_y(\varphi_i \circ \varphi_j^{-1}) \cdot v\big)$$

is a homeomorphism and even a diffeomorphism.

We have therefore defined a topology on TM which makes it a topological manifold with the atlas $(TU_i, \Phi_i)_{i \in I}$. As this atlas is smooth, TM is a smooth manifold of dimension $2 \dim M$. At this stage, it is important to remark that if M is a C^p manifold (with $p > 0$), then TM is a C^{p-1} manifold. This manifold is called the *tangent bundle* to M. We justify this name.

Proposition 3.24. *The canonical projection p from TM to M is a fibration.*

4.3. The Lie Algebra of a Lie Group

4.3.1. Basic Properties; The Adjoint Representation

For a Lie group, we have just seen that it is the same thing to have

– a left invariant vector field;

– a vector in the tangent space to the identity;

– a one-parameter subgroup.

In particular, every algebraic operation defined on one of these objects, such as the *bracket* for left invariant vector fields, can be transported to the others.

Definition 4.9. *A* Lie algebra *over a field K is a vector space L over K, equipped with a bilinear map from $L \times L$ to L, called the* bracket*, denoted* $[\ ,\ ]$*, such that*

i) $\forall X \in L$, $[X, X] = 0$.

ii) $\forall X, Y, Z \in L$, $[[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Examples

a) Any vector space equipped with the zero bracket is a Lie algebra. This is the only case (at least in characteristic not equal to 2) where a Lie algebra is commutative, as calculating $[X+Y, X+Y]$ shows that $[X, Y]+[Y, X] = 0$.

b) The results of Section 3.6 may be reformulated by saying that for every smooth manifold M, the vector space $C^\infty(TM)$ equipped with the Lie bracket is a Lie algebra (of infinite dimension, since $C^\infty(M)$ is already infinite dimensional).

c) By Proposition 4.4, the left invariant (or right invariant) vector fields on a Lie group form a finite-dimensional Lie algebra.

Definition 4.10. *A* morphism *of Lie algebras L and L' over the same field K is a linear map f from L to L' such that*

$$\forall X, Y \in L, \quad f([X, Y]) = [f(X), f(Y)].$$

If f is invertible, it is clear that f^{-1} is also a morphism. We then say that f is a Lie algebra isomorphism.

Example. Again by Proposition 4.4, $\mathcal{I}_*$ is an isomorphism between the algebras of left and right invariant vector fields on G.

By transporting the structure, the tangent space to the identity of a Lie group is equipped with a Lie algebra structure.

5.2. Multilinear Algebra

5.2.1. Tensor Algebra

Let E be a vector space over a field K. The *dual space* $\mathcal{L}(E,K) = E^*$ is the vector space of K-linear maps from E to K, also called *linear forms.* Suppose E has dimension n, and suppose $(e_i)_{1\leqslant i\leqslant n}$ is a basis. If

$$v = \sum_{i=1}^{n} v^i e_i$$

is the decomposition of a vector with respect to this basis, we denote by e^{i*} the linear form $v \mapsto v^i$, which associates to every vector its i-th coordinate. Then if $\alpha \in E^*$, we have

$$\alpha(v) = \sum_{i=1}^{n} v^i \alpha(e_i) = \sum_{i=1}^{n} \alpha(e_i) e^{i*}(v)$$

for all v. In other words the linear form α may be written as the linear combination

$$\alpha = \sum_{i=1}^{n} \alpha(e_i) e^{i*}.$$

In particular, $(e^{i*})_{1\leqslant i\leqslant n}$ is a basis of E^*, called the *dual basis* to $(e_i)_{1\leqslant i\leqslant n}$.

We use the *Einstein summation convention.* When we index a family of vectors or a vector field, we write a *lower* index. A good mnemonic is to think of vector fields ∂_i. When we index forms, we use an *upper* index, whether actual forms like e^{i*}, or their values on a vector such as the numbers v^i. When we decompose a vector (resp. a form) with respect to a basis, we place the indices of the coefficients in upper (resp. lower) position as we have just done. Physicists have profited from the convention that an expression where the same index appears both in upper and lower position as representing a sum over this index. For our part we will not omit the summation signs, but we will adopt the convention above for the placement of indices. This usage allows us to see at a glance whether we are working with vectors or forms.

Definition 5.1. *A* linear k-form *on E is any map*

$$L : \overbrace{E \times \cdots \times E}^{k \text{ times}} \longrightarrow K$$

such that the component functions

$$x_r \longmapsto L(x_1, \ldots, x_k)$$

are linear forms on E.

Examples

a) Consider the disk with center 0 and radius a in $\mathbf{R}^2$ equipped with the natural Euclidean norm. The boundary is the circle of center 0 and radius a. In a neighborhood of a point p on the circle, we take polar coordinates (r, θ). In these coordinates, the volume form $dx \wedge dy$ may be written $rdr \wedge d\theta$, and with the notations of Lemma 6.23, we can take $\varphi_1 = r - a, \varphi_2 = \theta$. The orientation of the circle is given by $d\theta$. In other words, $\theta \mapsto (a\cos\theta, a\sin\theta)$ preserves the orientation. We say that the circle is oriented counterclockwise.

b) We can reconsider the discussion for a circular annulus $C(a, b)$, where $a < b$. As a result of the above, the circle of radius b is oriented in the trigonometric sense. Conversely, in a neighborhood of the circle of radius a we must take $\varphi_1 = a - r$, thus $\varphi_2 = -\theta$. The orientation is thus the opposite to the trigonometric orientation.

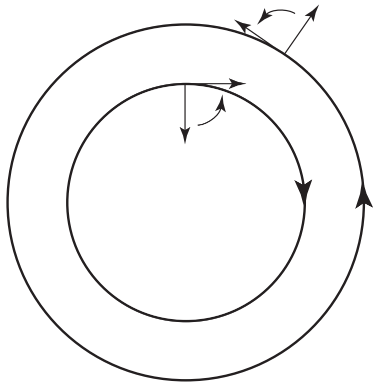

Figure 6.2: Oriented boundary of an annulus

In the same way, if D is a closed annulus of Euclidean space, ∂D has two connected components which are spheres with *opposite* orientations.

c) Figure 6.3 can be deduced in the same way as in b).

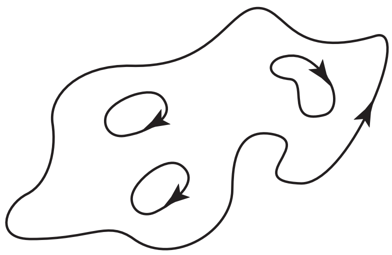

Figure 6.3: Another oriented boundary

d) Let M be a compact submanifold of codimension 1 in $\mathbf{R}^n$. Then by Alexander's theorem (cf. [Bredon 94]), $\mathbf{R}^n \setminus M$ has two connected components, one of which is bounded. The closure of the bounded component is a regular domain with boundary M. In particular, M is orientable. For $n = 2$, this result is called Jordan's theorem. It is already nontrivial (see [Berger-Gostiaux 88, 9.2]) or [Do Carmo 76, 5.7].

e) The case where $\dim D = 1$ merits special attention as the preceding proof does not apply directly. Now ∂D consists of a finite number of points (in fact 2 if D is connected). An "orientation" of a point is a choice of sign $\pm$ (the forms of degree 0 are functions which are constant here). A boundary point p is assigned the sign $+$ if in a neighborhood of p, D is defined by $x \leqslant 0$, where x is a local coordinate compatible with the orientation of D, and is assigned $-$ otherwise. Notice $\{b\} - \{a\}$ is the oriented boundary of $[a, b]$ in $\mathbf{R}$.

6.4.3. Stokes's Theorem in All of Its Forms

Theorem 6.25 (Stokes). *Let D be a regular domain of an oriented manifold M of dimension n, ∂D is oriented boundary, and let $\alpha \in \Omega^{n-1}(M)$. Then*

$$\int_{\partial D} \alpha_{|\partial D} = \int_D d\alpha.$$

Example. If D is the interval $[a, b] \subset \mathbf{R}$, then α is a function f, and with the preceding convention we have

$$\int_{\partial[a,b]} f = f(b) - f(a),$$

and we recover the fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

Proof. First, take an open set V containing D whose closure is compact. We can cover V by a finite number of domains of charts U_i such that if U_i intersects ∂D, then $U_i \cap \partial D = \{x, x^1 = 0\}$, the orientation of ∂D being given on $U_i \cap \partial D$ by $\varphi_i^*(dx^2 \wedge \cdots \wedge dx^n)$. If f_i is a partition of unity subordinate to this cover, we have $\alpha = \sum_{i \in I} f_i \alpha$, and it suffices to prove the result for the $\alpha_i = f_i \alpha$, which is to say for forms supported within the open subset U_i.

Thus let α be such that $\operatorname{Supp} \alpha \subset U_i$. There are three cases to consider:

1) If $\operatorname{Supp} \alpha \subset M \setminus D$, then α vanishes on ∂D, and $d\alpha$ vanishes on D. The result is then clear.

unit tangent vector, and $n(s)$ is the vector such that the frame $\big(\tau(s), n(s)\big)$ is positively oriented.

For a triangle formed by three C^2 arcs with angles $(\beta_i)_{1\leqslant i\leqslant 3}$, we then have

$$\beta_1 + \beta_2 + \beta_3 = \pi + \int_T k(s)\, ds$$

or, by introducing the exterior angles α_i,

$$\alpha_1 + \alpha_2 + \alpha_3 + \int_T k(s)\, ds = 2\pi.$$

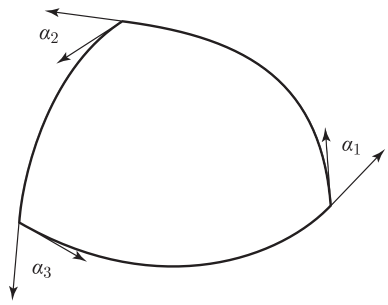

Figure 8.1: Gauss-Bonnet for a triangle

In case there is no angle at a point, $\alpha_i = 0$ and $\beta_i = \pi$, we obtain

$$\int_T k(s)\, ds = 2\pi.$$

This is the *Umlaufsatz* or theorem of turning tangents, see [Berger-Gostiaux 88, 9.5] or [Chavel 83, 4.6].

These results are natural: knowing that $k(s) = \varphi'(s)$, where φ is the angle $\tau(s)$ forms with a fixed vector, this simply says that the unit tangent vector turns exactly 2π. Natural does not mean easy to prove however, as in the example of Jordan's theorem which ensures that the complement of a simple closed curve has two connected components.

This formula was generalized by C.-F. Gauss (who did not publish it) and by P.-O. Bonnet to triangles constrained to a surface. The curves which replace straight lines are geodesics, which is to say curves that minimize length, and the function $k(s)$, whose vanishing characterizes geodesics, is the algebraic measure of the orthogonal projection of the acceleration onto the tangent plane. (See [Do Carmo 76, p. 248].)

6*. *Forms invariant under a group*

a) Use the fact that

$$\omega = i_X(dx^0 \wedge \cdots \wedge dx^n), \text{ where } X \text{ is the radial vector field.}$$

To see that Ω is the only form of degree n which is invariant under $Sl(n+1, \mathbf{R})$, note first that $Sl(n+1, \mathbf{R})$ acts transitively on $\mathbf{R}^{n+1} \setminus \{0\}$, and such a form is determined on $\mathbf{R}^{n+1} \setminus \{0\}$ by its value at $e_0 = (1, 0, \ldots, 0)$ for example. We thus reduce to showing that $e^{1*} \wedge \cdots \wedge e^{n*}$ is the only n-linear alternating form (up to a factor) which is invariant under the subgroup of $Sl(n+1, \mathbf{R})$ which fixes e_0 (here we denoted the basis dual to the canonical basis of $\mathbf{R}^{n+1}$ by $(e^{i*})_{0 \leqslant i \leqslant n}$).

b) Take inspiration from 12, c4) further below.

7. The primitive of

$$\alpha = \sum_{1 \leqslant i < j \leqslant n} \alpha_{ij} dx^i \wedge dx^j$$

thus obtained is

$$\beta = \sum_{1 \leqslant i < j \leqslant n} \left(\int_0^1 \alpha_{ij}(ux)\, du \right) (x^i\, dx^j - x^j\, dx^i).$$

8. *Forms invariant under a Lie group*

b) It suffices to calculate $d\omega(V_0, \ldots, V_p)$ for left invariant vector fields by applying Theorem 5.24.

c) We have $dX^{-1} = -X^{-1} dX X^{-1}$ (compare to the case of the linear group seen in Section 1.2). If $\Omega = X^{-1} dX$, we have

$$d\Omega + \Omega \wedge \Omega = 0,$$

where the matrix $\Omega \wedge \Omega$ is defined by

$$(\Omega \wedge \Omega)_i^j = \sum_k (\Omega)_i^k \wedge (\Omega)_k^j.$$

If U and V are two left invariant vector fields, we deduce that

$$d\Omega([U, V]) = (\Omega \wedge \Omega)(U, V).$$

We discover the expression for the bracket by evaluating each side at the identity element.

d) Restrict the matrices $X^{-1} dX$ and $(dX) X^{-1}$ to G. The vector space of left invariant forms of degree 1 is generated by $a^{-1} da$ and $a^{-1} db$, and that of