MULTIDIMENSIONAL MINIMIZING SPLINES

THEORY AND APPLICATIONS

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Multidimensional Minimizing Splines is a book for post-graduates. It follows a graduate level title : *Approximation hilbertienne - Splines, ondelettes, fractales* by M. ATTÉIA and J. GACHES, published in the Grenoble Sciences' French series (Collection Grenoble Sciences, EDP Sciences, 1999, 158 p.).

Grenoble Sciences is supported by the French Ministry of Higher Education, the Ministry of Research and the « Région Rhône-Alpes ».

EXTRAITS

Proof – Replacing Ω by Ω' and taking Proposition 3.1 into account, one resumes the proof of Theorem V–3.1. □

Remark 3.2 – Reasoning as in the proof of Theorem V–2.1, one can show that the interpolating and smoothing D^m-splines over Ω' relative to ρ belong to the space $C^{2m-n-1-\mu}(\Omega')$. Therefore, in particular, σ and σ_ε belong to $C^1(\Omega')$ if $m = 2$, $n = 2$ and $\mu = 0$, and to $C^2(\Omega')$ if $m = 3$, $n = 2$ and $\mu = 1$. □

Remark 3.3 – Suppose, in this section, that Σ is a set of linear forms either of type $v \mapsto v(a)$, with $a \in \overline{\Omega}$, or of types (3.1) or (3.2), with $|\alpha| = 1$. Replace $H^m(\Omega')$ by the space $V = H^m(\Omega') \cap C^0(\overline{\Omega})$. Then, taking Theorem 2.4 into account, one defines in the same way the V-interpolating (resp. V-smoothing) D^m-spline relative to ρ and β (resp. ρ, β and ε). □

4. *DISCRETE D^m-SPLINES*

In order to simplify the exposition, we suppose from now on that Ω is a *polyhedral* subset of $\mathbb{R}^n$ (this assumption is verified in the applications). On the other hand, we suppose that the closure of the discontinuity set F is a finite union of (closed) faces of polyhedrons in $\mathbb{R}^n$, that m is any positive integer and we denote by k an integer equal to 1 or 2. We write $\Omega' = \Omega \setminus \overline{F}$, we keep the notations A, Σ, μ, N and ρ of Section 3 and we suppose that (3.4) is verified.

Let $\mathbb{H}$ be a bounded subset of $(0, +\infty)$ such that $0 \in \overline{\mathbb{H}}$. For any $h \in \mathbb{H}$, suppose we are given

- a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ by means of n-simplices K with diameters $h_K \leq h$ and pairwise disjoint interiors $\overset{\circ}{K}$, such that

$$\forall K \in \mathcal{T}_h, \ \overset{\circ}{K} \cap F = \emptyset, \tag{4.1}$$

$$\left| \begin{array}{l} \text{any face of a } n\text{-simplex } K \in \mathcal{T}_h \text{ is either the face of another } n\text{-} \\ \text{simplex in } \mathcal{T}_h, \text{ or a part of } \partial\Omega, \text{ or a part of } \overline{F} \end{array} \right. \tag{4.2}$$

  (cf. Figure 2),

- a finite element space V_h, constructed on $\mathcal{T}_h$, such that

$$V_h \text{ is a finite-dimensional subspace of } H^m(\Omega') \cap C^k_F(\Omega'). \tag{4.3}$$

Remark 4.1 – Let k' be the class of the generic finite element of the space V_h. Then, hypothesis (4.3) implies that $k \leq k'$ and that $m \leq k' + 1$, so that the inclusions $V_h \subset C^k_F(\Omega')$ and $V_h \subset H^m(\Omega')$, respectively, are obtained. When dealing with real problems, one takes $k' = k$, for reasons of cost.

For the problem of approximating surfaces from Lagrange or first order Hermite data, we have $n = 2$ and so, taking account of hypothesis (5.2), needed in study of the convergence, the usual choices are $k' = 1$ and $m = 2$, if $k = 1$, and $k' = 2$ and $m = 3$,

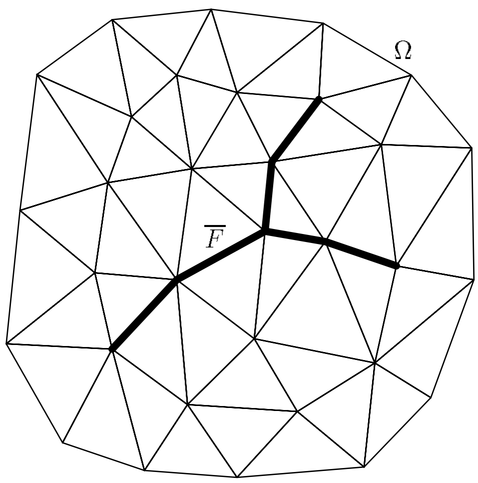

Figure 2: Example of triangulation of the set $\overline{\Omega}$.

if $k = 2$ (see Section VIII–2). Notice that we must use *triangular* finite elements, because F may be any polygonal set. For examples of generic finite elements, the reader is referred to Section VIII–3. □

Remark 4.2 – Let us detail how to obtain a finite element space V_h satisfying (4.3). In a first step, one follows the usual process in the Finite Element Method, without taking F into account, in order to construct a finite element space V_h^* such that $V_h^* \subset H^m(\Omega) \cap C^k(\overline{\Omega})$. Let $w_1^*, \dots, w_{M^*}^*$ be the basis functions of V_h^*. For $i = 1, \dots, M^*$, let $b_i \in \overline{\Omega}$ be the node with which w_i^* is associated and let γ_i be the number of connected components of $(\operatorname{supp} w_i^*) \setminus \overline{F}$, whose respective closures are denoted by $U_i^1, \dots, U_i^{\gamma_i}$ (cf. Figure 3). It is obvious that $\gamma_i > 1$ only if b_i belongs to F (it may happen, however, that $\gamma_i = 1$ for some nodes $b_i \in \partial\Omega \cap F$).

Now, let $W = \{ w_i^* \chi_{U_i^j} \mid i = 1, \dots, M^*, \ j = 1, \dots, \gamma_i \}$, where $\chi_{U_i^j}$ is the characteristic function of U_i^j. It is clear that W is a finite family of linearly independent functions of $H^m(\Omega') \cap C_F^k(\Omega')$. Then, the space V_h is just the linear space spanned by W. Let us observe that the sets U_i^j are the supports of the functions in W and so the supports of the basis functions of V_h. □

According to (4.3) and since $\mu \leq k$, we can define on V_h, for any $h \in \mathbb{H}$, the mapping $[\![\cdot]\!]_{m,\Omega'}$, introduced in (3.5). It follows from (3.4) that $[\![\cdot]\!]_{m,\Omega'}$ is a norm on V_h. Of course, endowed with this norm, V_h is a Hilbert space, because it is finite-dimensional.

Let $\beta \in \mathbb{R}^N$. For any $h \in \mathbb{H}$, we define the vector space

$$\mathcal{K}_{0h} = \{ v_h \in V_h \mid \rho v_h = 0 \}$$

and the affine linear variety

$$\mathcal{K}_h = \{ v_h \in V_h \mid \rho v_h = \beta \}.$$

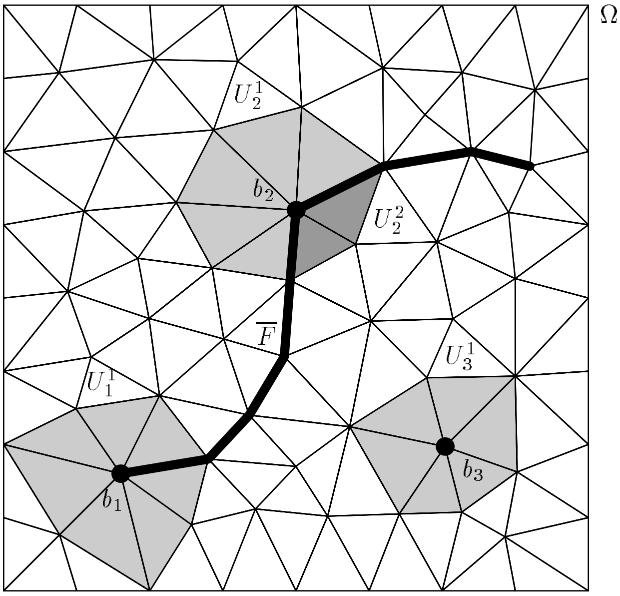

Figure 3: Example of sets U_i^j associated with three nodes b_i (cf. Remark 4.2). Let us observe that $\gamma_1 = \gamma_3 = 1$, whereas $\gamma_2 = 2$.

Then we consider the problem: find σ_h solution of

$$\begin{cases} \sigma_h \in \mathcal{K}_h, \\ \forall v_h \in \mathcal{K}_h, \ |\sigma_h|_{m,\Omega'} \leq |v_h|_{m,\Omega'}. \end{cases} \tag{4.4}$$

Any solution σ_h of (4.4), if any exists, is called V_h*-discrete interpolating* D^m*-spline relative to* ρ *and* β.

***Theorem* 4.1** – *Suppose that hypotheses* (3.4)*,* (4.1)*,* (4.2) *and* (4.3) *are verified. Moreover, suppose that*

$$\forall h \in \mathbb{H}, \ \Sigma \subset \Sigma_h, \tag{4.5}$$

where Σ_h *denotes the set of degrees of freedom of* V_h*. Then, for any* $h \in \mathbb{H}$*, problem* (4.4) *has a unique solution* σ_h*, characterized by*

$$\begin{cases} \sigma_h \in \mathcal{K}_h, \\ \forall w_h \in \mathcal{K}_{0h}, \ (\sigma_h, w_h)_{m,\Omega'} = 0. \end{cases} \tag{4.6}$$

Proof – Let us show that $\mathcal{K}_h$ is nonempty. Let us denote by $\phi_1, \ldots, \phi_N$ the elements of Σ. Let M be the dimension of V_h and let $w_1, \ldots, w_M$ be the basis functions of V_h, numbered so that, for any $j = 1, \ldots, N$, $\phi_j(w_j) = 1$ (which, by (4.5), we are able to do). Then, the function $v_h = \sum_{j=1}^N \beta_j w_j$ belongs to $\mathcal{K}_h$. Thus, the closed convex nonempty subset $\mathcal{K}_h$ of V_h has a unique element σ_h of minimal norm $[\![\cdot]\!]_{m,\Omega'}$, characterized by (4.6). □

When (3.3) is verified, it is clear that (4.4) (resp. (4.6)) constitutes a discretization of (3.6) (resp. (3.7)).

Remark 4.3 – With the notations of the previous proof and under hypothesis (4.5), the solution σ_h of (4.4) can be written as

$$\sigma_h = \sum_{j=1}^{N} \beta_j w_j + \sum_{j=N+1}^{M} \alpha_j w_j,$$

with $\alpha_j \in \mathbb{R}$, for $j = N+1, \ldots, M$. Reasoning as in Remark VI–2.1, we see that the unknown coefficients α_j are the solution of the linear system

$$\sum_{j=N+1}^{M} (w_j, w_i)_{m,\Omega'} \alpha_j = -\sum_{j=1}^{N} \beta_j (w_j, w_i)_{m,\Omega'}, \quad N+1 \leq i \leq M,$$

whose matrix is *regular*. □

For any $\varepsilon > 0$ and any $h \in \mathbb{H}$, we now consider the problem: find $\sigma_{\varepsilon h}$ verifying

$$\begin{cases} \sigma_{\varepsilon h} \in V_h, \\ \forall v_h \in V_h, \ J_\varepsilon(\sigma_{\varepsilon h}) \leq J_\varepsilon(v_h), \end{cases} \tag{4.7}$$

where J_ε denotes the functional introduced in (3.10) (let us observe that, independently of any condition on m, such as (3.3), J_ε is defined, in fact, on $H^m(\Omega') \cap C_F^\mu(\Omega')$ and hence on V_h).

***Theorem* 4.2** – *Under hypotheses* (3.4), (4.1), (4.2) *and* (4.3), *for any* $h \in \mathbb{H}$, *problem* (4.7) *has a unique solution* $\sigma_{\varepsilon h}$, *called* V_h-discrete smoothing D^m-spline relative to ρ, β and ε, *which is also the unique solution of the problem: find* $\sigma_{\varepsilon h}$ *such that*

$$\begin{cases} \sigma_{\varepsilon h} \in V_h, \\ \forall v_h \in V_h, \ \langle \rho\sigma_{\varepsilon h}, \rho v_h \rangle + \varepsilon(\sigma_{\varepsilon h}, v_h)_{m,\Omega'} = \langle \beta, \rho v_h \rangle. \end{cases} \tag{4.8}$$

Proof – Taking into account that V_h, endowed with the norm $[\![\cdot]\!]_{m,\Omega'}$, is a Hilbert space, the proof is similar to that of Theorem V–3.1. □

When (3.3) is verified, (4.7) (resp. (4.8)) is clearly a discretization of (3.11) (resp. (3.12)).

Remark 4.4 – Let us write $\sigma_{\varepsilon h} = \sum_{j=1}^{M} \alpha_j w_j$, where $w_1, \ldots, w_M$ denote the basis functions of V_h, and let us introduce the matrices

$$\mathcal{A} = \big(\phi_i(w_j)\big)_{1 \leq i \leq N,\, 1 \leq j \leq M}$$

and

$$\mathcal{R} = \big((w_j, w_i)_{m,\Omega'}\big)_{1 \leq i,j \leq M}.$$

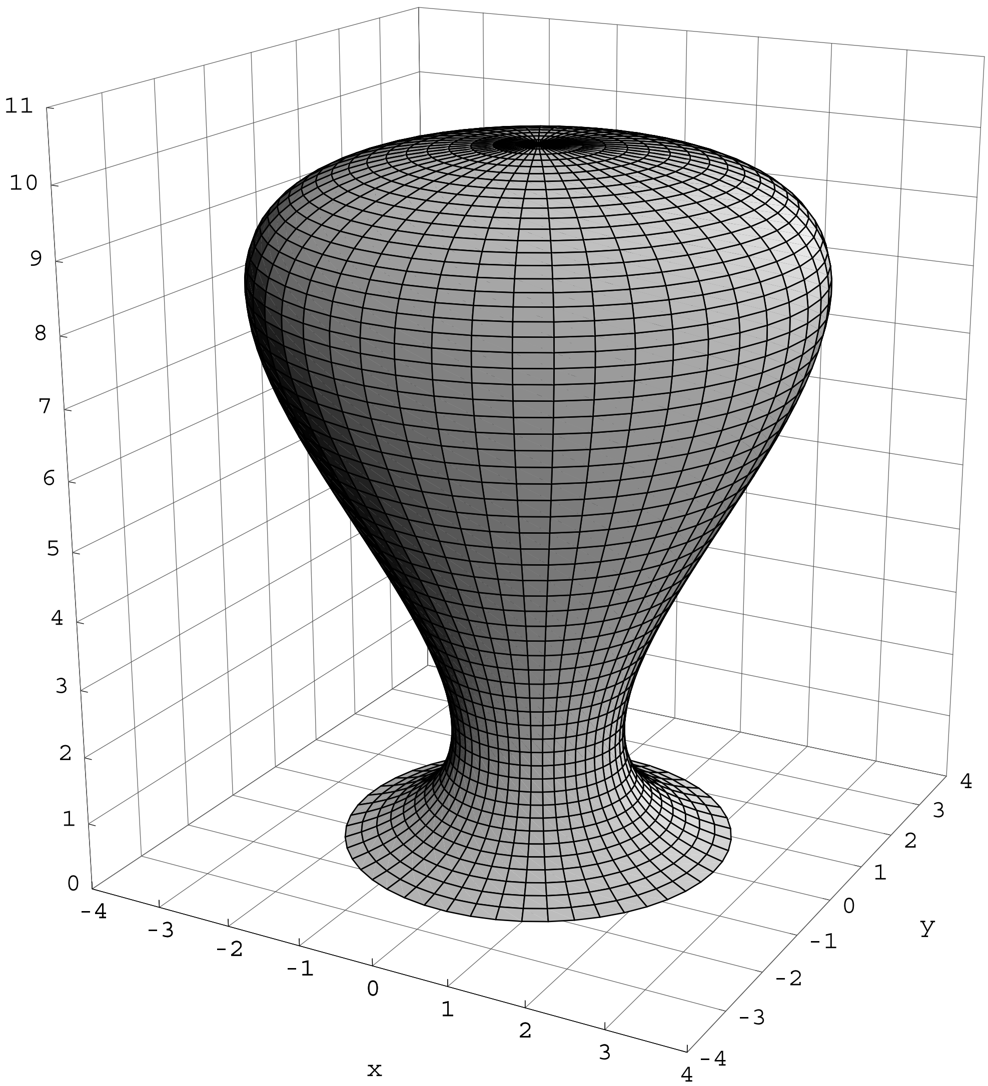

Figure 11: Example 3. Surface $\mathcal{S}$.

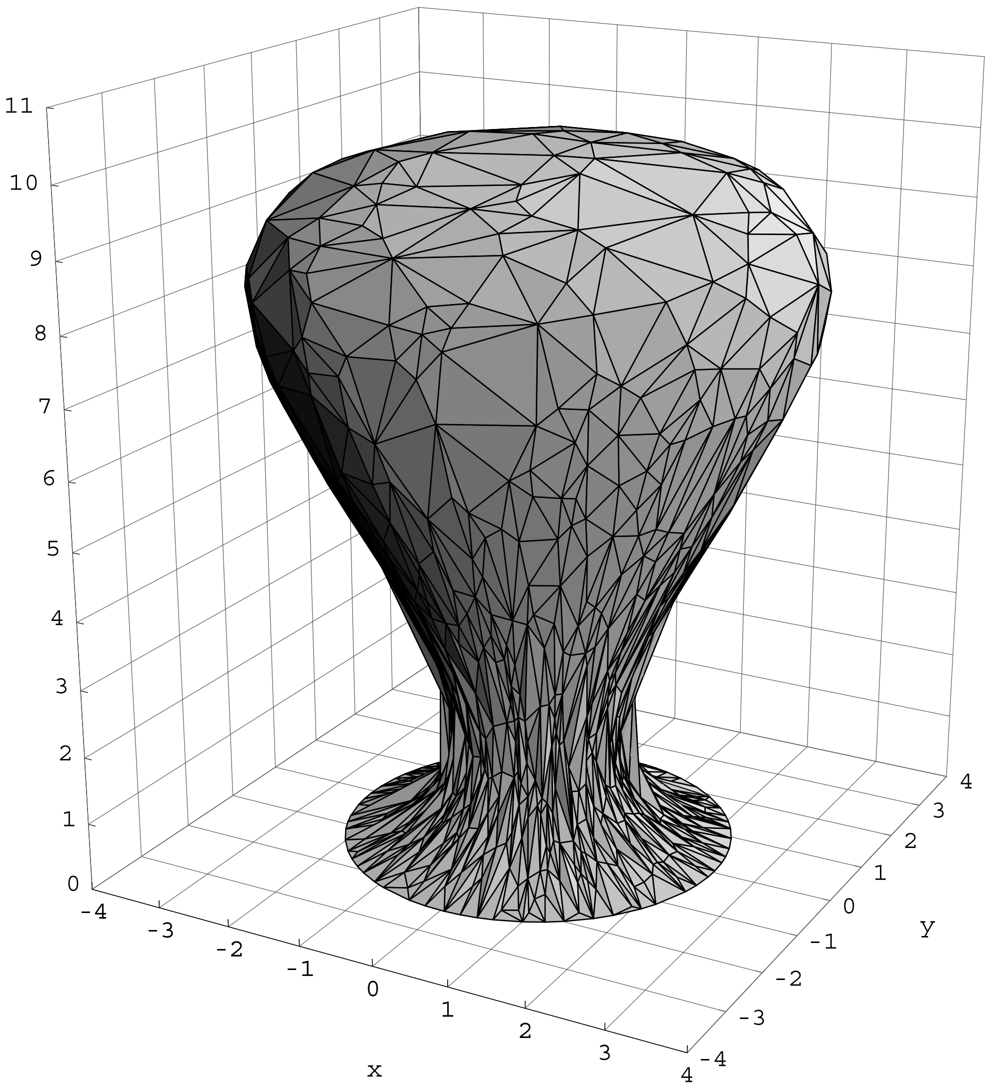

Figure 12: Example 3. Surface triangulation $\mathbb{T}$.

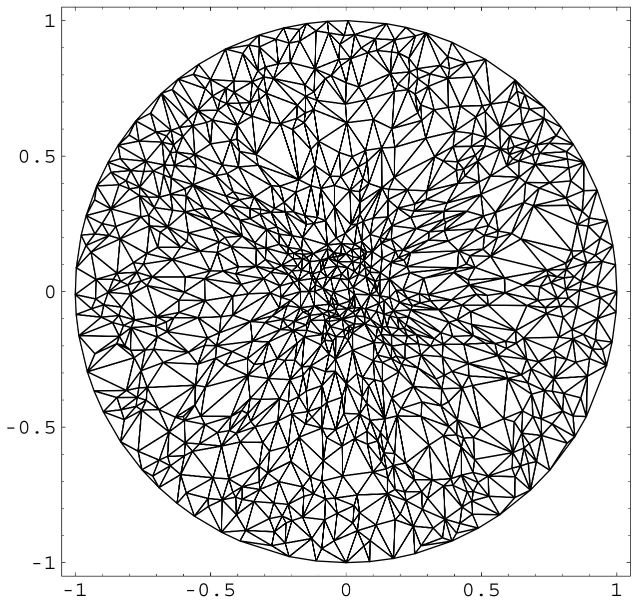

Figure 13: Example 3. Planar triangulation yielded by the shape-preserving parametrization method.

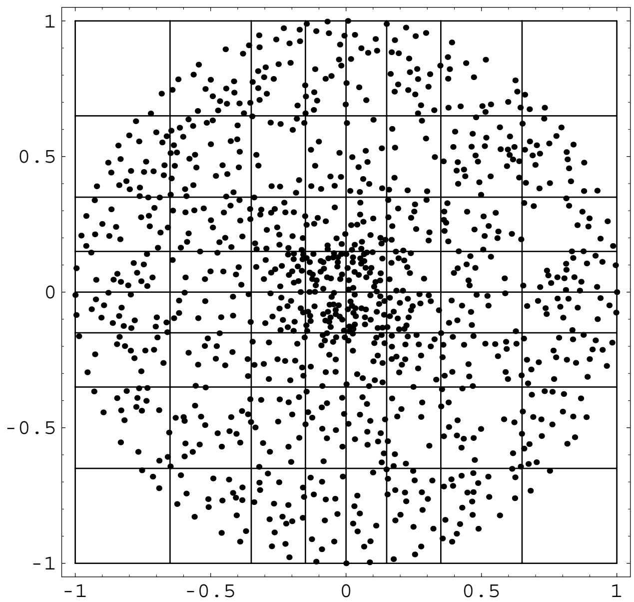

Figure 14: Example 3. Set A of parameter points and triangulation $\widetilde{\mathcal{T}}_h$.

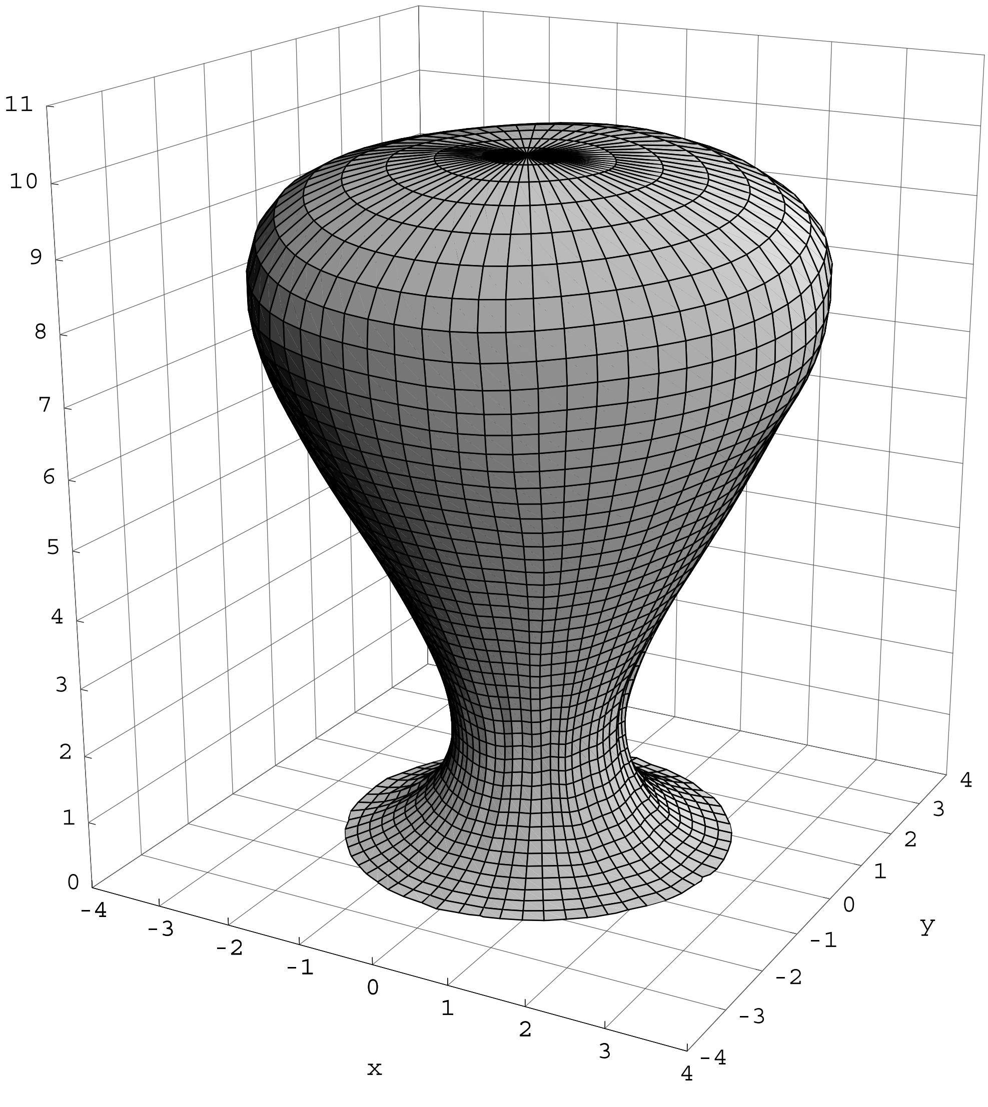

Figure 15: Example 3. Trace of the restriction to Ω of the X_h-discrete smoothing D^2-spline $\sigma_{\varepsilon h}$ relative to A, $\mathcal{P}$ and $\varepsilon = 10^{-5}$. Relative error: $r(\mathcal{S}) = 0.00353369$.