Chaouqi Misbah

Complex Dynamics and Morphogenesis

An introduction to nonlinear science

Springer

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