Contents

Foreword v

Contents ix

Synoptic Tables of the problems 1

Chapter 1. The Lagrangian formulation 9

Summary 9

1.1. Generalized coordinates 9

1.2. Lagrange's equations 10

1.3. Generalized forces 12

1.4. Lagrange multipliers 13

Problem statements 14

1.1. The wheel jack 14

1.2. The sling 15

1.3. Rope slipping on a table 16

1.4. Reaction force for a bead on a hoop 16

1.5. Huygens pendulum 17

1.6. Cylinder rolling on a moving tray 18

1.7. Motion of a badly balanced cylinder 18

1.8. Free axle on a inclined plane 19

1.9. The turn indicator 21

1.10. An experiment to measure the rotational velocity of the Earth 22

1.11. Generalized inertial forces 23

Problem solutions 14

1.1. The wheel jack 24

1.2. The sling 26

1.3. Rope slipping on a table 27

1.4. Reaction force for a bead on a hoop 28

1.5. Huygens pendulum 31

1.6. Cylinder rolling on a moving tray 33

1.7. Motion of a badly balanced cylinder 35

1.8. Free axle on a inclined plane 39

1.9. The turn indicator 43

1.10. An experiment to measure the rotational velocity of the Earth 46

1.11. Generalized inertial forces 48

CHAPTER 2. LAGRANGIAN SYSTEMS 51

Summary 51

2.1. Generalized potential 51

2.2. Lagrangian system 52

2.3. Constants of the motion 53

2.4. Two-body system with central force 55

2.5. Small oscillations 56

Problem statements 14

2.1. Disc on a movable inclined plane 57

2.2. Painlevé's integral 58

2.3. Application of Noether's theorem 58

2.4. Foucault's pendulum 59

2.5. Three-particle system 61

2.6. Vibration of a linear triatomic molecule: the "soft" mode 63

2.7. Elastic transversal waves in a solid (F waves) 64

2.8. Lagrangian in a rotating frame 65

2.9. Particle drift in a constant electromagnetic field 66

2.10. The Penning trap 67
2.11. Equinox precession 68
2.12. Flexion vibration of a blade 71
2.13. Solitary waves 73
2.14. Vibrational modes of an atomic 75
Problem solutions 76
2.1. Disc on a movable inclined plane 76
2.2. Painlevé's integral 77
2.3. Application of Noether's theorem 78
2.4. Foucault's pendulum 79
2.5. Three-particle system 82
2.6. Vibration of a linear triatomic molecule: the "soft" mode 86
2.7. Elastic transversal waves in a solid (F waves) 88
2.8. Lagrangian in a rotating frame 89
2.9. Particle drift in a constant electromagnetic field 91
2.10. The Penning trap 94
2.11. Equinox precession 97
2.12. Flexion vibration of a blade 102
2.13. Solitary waves 105
2.14. Vibrational modes of an atomic 107

CHAPTER 3. HAMILTON'S PRINCIPLE 111
Summary 111
3.1. Statement of the principle 111
3.2. Action functional 112
3.3. Action and field theory 112
3.4. Some well known actions 113
3.5. The calculus of variations 114
Problem statements 14
3.1. The Lorentz force 116

3.2. Relativistic particle in a central force field 117
3.3. Principle of least action? 118
3.4. Minimum or maximum action? 119
3.5. Is there only one solution which makes the action stationary? 120
3.6. The principle of Maupertuis 121
3.7. Fermat's principle 122
3.8. The skier strategy 122
3.9. Free motion on an ellipsoid 123
3.10. Minimum area for a fixed volume 124
3.11. The form of soap films 125
3.12. Laplace's law for surface tension 127
3.13. Chain of pendulums 128
3.14. Wave equation for a flexible blade 128
3.15. Precession of Mercury's orbit 128
Problem solutions 131
3.1. The Lorentz force 131
3.2. Relativistic particle in a central force field 132
3.3. Principle of least action? 135
3.4. Minimum or maximum action? 137
3.5. Is there only one solution which makes the action stationary? 138
3.6. The principle of Maupertuis 141
3.7. Fermat's principle 144
3.8. The skier strategy 146
3.9. Free motion on an ellipsoid 150
3.10. Minimum area for a fixed volume 152
3.11. The form of soap films 154
3.12. Laplace's law for surface tension 158
3.13. Chain of pendulums 160

3.14. Wave equation for a flexible blade 161
3.15. Precession of Mercury's orbit 162

CHAPTER 4. HAMILTONIAN FORMALISM 165
Summary 165
4.1. Generalized momentum 165
4.2. Hamilton's function 166
4.3. Hamilton's equations 167
4.4. Liouville's theorem 167
4.5. Autonomous one-dimensional systems 168
4.6. Periodic one-dimensional Hamiltonian systems 169
Problem statements 171
4.1. Electric charges trapped in conductors 171
4.2. Symmetry of the trajectory 171
4.3. Hamiltonian in a rotating frame 172
4.4. Identical Hamiltonian flows 173
4.5. The Runge-Lenz vector 173
4.6. Quicker and more ecologic than a plane 174
4.7. Hamiltonian of a charged particle 176
4.8. The first integral invariant 177
4.9. What about non-autonomous systems? 178
4.10. The reverse pendulum 178
4.11. The Paul trap 180
4.12. Optical Hamilton's equations 181
4.13. Application to billiard balls 183
4.14. Parabolic double well 184
4.15. Stability of circular trajectories in a central potential 185
4.16. The bead on the hoop 186
4.17. Trajectories in a central force field 188

Problem solutions 188
4.1. Electric charges trapped in conductors 188
4.2. Symmetry of the trajectory 190
4.3. Hamiltonian in a rotating frame 192
4.4. Identical Hamiltonian flows 194
4.5. The Runge-Lenz vector 195
4.6. Quicker and more ecologic than a plane 198
4.7. Hamiltonian of a charged particle 200
4.8. The first integral invariant 204
4.9. What about non-autonomous systems? 206
4.10. The reverse pendulum 207
4.11. The Paul trap 211
4.12. Optical Hamilton's equations 214
4.13. Application to billiard balls 216
4.14. Parabolic double well 219
4.15. Stability of circular trajectories in a central potential 222
4.16. The bead on the hoop 224
4.17. Trajectories in a central force field 228

CHAPTER 5. HAMILTON-JACOBI FORMALISM 233
Summary 233
5.1. The action function 233
5.2. Reduced action 234
5.3. Maupertuis' principle 235
5.4. Jacobi's theorem 236
5.5. Separation of variables 236
5.6. Huygens' construction 238
Problem statements 239
5.1. How to manipulate the action and the reduced action 239
5.2. Action for a one-dimensional harmonic oscillator 241

5.3. Motion on a surface and geodesic 241
5.4. Wave surface for free fall 242
5.5. Peculiar wave fronts 243
5.6. Electrostatic lens 243
5.7. Maupertuis' principle with an electromagnetic field 245
5.8. Separable Hamiltonian, separable action 246
5.9. Stark effect 247
5.10. Orbits of Earth's satellites 248
5.11. Phase and group velocities 251
Problem solutions 252
5.1. How to manipulate the action and the reduced action 252
5.2. Action for a one-dimensional harmonic oscillator 258
5.3. Motion on a surface and geodesic 260
5.4. Wave surface for free fall 261
5.5. Peculiar wave fronts 264
5.6. Electrostatic lens 265
5.7. Maupertuis' principle with an electromagnetic field 268
5.8. Separable Hamiltonian, separable action 270
5.9. Stark effect 271
5.10. Orbits of Earth's satellites 275
5.11. Phase and group velocities 279

Chapter 6. Integrable systems 281
Summary 281
6.1. Basic notions 281
6.1.1. Some definitions 281
6.1.2. Good coordinates: The angle–action variables 283
6.2. Complements 286
6.2.1. Building the angle variables 286

6.2.2. Flow/Poisson bracket/Involution 287
6.2.3. Criterion to obtain a canonical transformation 288
Problem statements 289
6.1. Expression of the period for a one-dimensional motion 289
6.2. One-dimensional particle in a box 290
6.3. Ball bouncing on the ground 290
6.4. Particle in a constant magnetic field 291
6.5. Actions for the Kepler problem 292
6.6. The Sommerfeld atom 293
6.7. Energy as a function of actions 294
6.8. Invariance of the circulation under a continuous deformation 296
6.9. Ball bouncing on a moving tray 297
6.10. Harmonic oscillator with a variable frequency 298
6.11. Choice of the momentum 298
6.12. Invariance of the Poisson bracket under a canonical transformation 299
6.13. Canonicity for a contact transformation 299
6.14. One-dimensional free fall 300
6.15. One-dimensional free fall again 301
6.16. Scale dilation as a function of time 301
6.17. From the harmonic oscillator to Coulomb's problem 302
6.18. Generators for fundamental transformations 303
Problem solutions 305
6.1. Expression of the period for a one-dimensional motion 305
6.2. One-dimensional particle in a box 306
6.3. Ball bouncing on the ground 308
6.4. Particle in a constant magnetic field 310
6.5. Actions for the Kepler problem 314
6.6. The Sommerfeld atom 316
6.7. Energy as a function of actions 318

6.8. Invariance of the circulation under a continuous deformation 322
6.9. Ball bouncing on a moving tray 324
6.10. Harmonic oscillator with a variable frequency 324
6.11. Choice of the momentum 325
6.12. Invariance of the Poisson bracket under a canonical transformation 326
6.13. Canonicity for a contact transformation 327
6.14. One-dimensional free fall 329
6.15. One-dimensional free fall again 330
6.16. Scale dilation as a function of time 332
6.17. From the harmonic oscillator to Coulomb's problem 333
6.18. Generators for fundamental transformations 336

CHAPTER 7. QUASI-INTEGRABLE SYSTEMS 341
Summary 341
7.1. Introduction 341
7.2. Perturbation theory 342
7.3. Canonical perturbation theory 342
7.4. Adiabatic invariants 345
Problem statements 347
7.1. Limits of the perturbative expansion 347
7.2. Non-canonical versus canonical perturbative expansion 347
7.3. First canonical correction for the pendulum 348
7.4. Beyond the first order correction 349
7.5. Adiabatic invariant in an elevator 350
7.6. Adiabatic invariant and adiabatic relaxation 351
7.7. Charge in a slowly varying magnetic field 352
7.8. Illuminations concerning the aurora borealis 354
7.9. Bead on a rigid stem: Hannay's phase 356

Problem solutions 358
7.1. Limits of the perturbative expansion 358
7.2. Non-canonical versus canonical perturbative expansion 361
7.3. First canonical correction for the pendulum 363
7.4. Beyond the first order correction 367
7.5. Adiabatic invariant in an elevator 370
7.6. Adiabatic invariant and adiabatic relaxation 372
7.7. Charge in a slowly varying magnetic field 375
7.8. Illuminations concerning the aurora borealis 379
7.9. Bead on a rigid stem: Hannay's phase 382

CHAPTER 8. FROM ORDER TO CHAOS FORMULATION 385
Summary 385
8.1. Introduction 385
8.2. The model of the kicked rotor 386
8.3. Poincaré's sections 388
8.4. The rotor for a null perturbation 388
8.5. Poincaré's sections for the kicked rotor 390
8.6. How to recognize fixed points 393
8.7. Separatrices/Homocline points/chaos 394
8.8. Complements 395
Problem statements 396
8.1. Disappearance of resonant tori 396
8.2. Continuous fractions or how to play with irrational numbers 396
8.3. Properties of the phase space of the standard mapping 398
8.4. Bifurcation of the periodic trajectory 1:1 for the standard mapping 398
8.5. Chaos–ergodicity : a slight difference 399
8.6. Acceleration modes: a curiosity of the standard mapping ... 401
8.7. Demonstration of a kicked rotor? 401
8.8. Anosov's mapping (or Arnold's cat) 403

8.9. Fermi's accelerator 405
8.10. Damped pendulum and standard mapping 407
8.11. Stability of periodic orbits on a billiard table 409
8.12. Lagrangian points: Jupiter's Greeks and Trojans 412
Problem solutions 415
8.1. Disappearance of resonant tori 415
8.2. Continuous fractions or how to play with irrational numbers 417
8.3. Properties of the phase space of the standard mapping 418
8.4. Bifurcation of the periodic trajectory 1:1 for the standard mapping 419
8.5. Chaos–ergodicity : a slight difference 423
8.6. Acceleration modes: a curiosity of the standard mapping ... 425
8.7. Demonstration of a kicked rotor? 427
8.8. Anosov's mapping (or Arnold's cat) 432
8.9. Fermi's accelerator 438
8.10. Damped pendulum and standard mapping 443
8.11. Stability of periodic orbits on a billiard table 447
8.12. Lagrangian points: Jupiter's Greeks and Trojans 450

BIBLIOGRAPHY 457

INDEX 461